Oscillator model for the relativistic fermion-boson system

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Abstract

The solvable quantum mechanical model for the relativistic two-body system composed of spin-1/2 and spin-0 particles is constructed. The model includes the oscillator-type interaction through a combination of Lorentz-vector and -tensor potentials. The analytical expressions for the wave functions and the order of the energy levels are discussed.

1 Introduction

Relativistic oscillator models for the spin-1/2 particles are a subject of rather broad interest nowadays. Apart from the academic interest in equations with exact solutions, the relativistic oscillator may serve as the first approximation to description of the more realistic systems in nuclear and particle physics. In contrast to the non-relativistic case, within the framework of the Dirac equation, there exist various possibilities for introducing the oscillator interaction, such as the mixture of the quadratic in coordinate vector and scalar potentials with the equal magnitude and equal or opposite signs [1–3], or the use of the non-minimal coupling scheme (the so-called Dirac oscillator) [4,5]. Note that the Dirac equation with the oscillator interaction has been applied recently for explaining the pseudospin symmetry in nuclei [6,7].

However, for the careful description of the relativistic effects in the two-body problem it is needed to go beyond the one-particle approximation. For this end, the Breit-type equations [8–11], the Barut method [12,13] and the relativistic quantum mechanics with constraints [14–17] are usually used. With these methods, the models for the relativistic two-boson [14,18] and two-fermion [19,20] oscillators have been offered.

Recently, a new approach to the relativistic two-body problem, based on the extension of the $SL(2,C)$ group to the $Sp(4,C)$ one, has been proposed [21]. This approach has been applied to construct the wave equation for the two-body system consisted of the spin-1/2 and spin-0 particles with interaction described by means of the Lorentz-vector and Lorentz-tensor potentials [22].

The goal of this work is to find, in addition to the mentioned two-boson and two-fermion oscillator models, an exact oscillator-like solution to the obtained equation [22] describing the fermion-boson system. In the lack of the Lorentz-vector interaction, this solution can be regarded as the generalization of the model of the one-body Dirac oscillator [4] to the fermion-boson case.

2 Wave equation with the extension of the $SL(2,C)$ group

In this Section, for completeness of the discussion let us briefly consider the wave equation for a fermion-boson system derived with the extension of the $SL(2,C)$ group.

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It is well known that the homogeneous Lorentz group $SO(1,3)$ is covered by the symplectic $Sp(2,C) \equiv SL(2,C)$ group. As a consequence, the relativistic field theory in the four-dimensional space-time can equivalently be formulated entirely within the framework of $Sp(2,C)$ Weyl spinor formalism [23]. It has been shown [21] that the extension of the $Sp(2,C)$ group to the $Sp(4,C)$ one permits us to develop a procedure of constructing relativistic wave equations for the two-body systems.

Following Ref. [22], we consider the system consisted of a spin-1/2 fermion and a spin-0 boson which interact with each other by virtue of the Lorentz-vector and Lorentz-tensor potentials. The wave function of this system is represented by a Dirac spinor or, in our treatment, by two $Sp(4,C)$ Weyl spinors φ and $\bar{\chi}$ and the corresponding wave equation has the Dirac-like form

$$\begin{aligned}
&[I \otimes \sigma^m (w_m + A_m) + \tau^1 \otimes \sigma^m (p_m + B_m)]\bar{\chi} \\
&= (m_+ + \tau^1 \otimes I m_- - \mathrm{i} I \otimes \sigma^m \tilde{\sigma}^n C_{mn} - \mathrm{i}\tau^1 \otimes \sigma^m \tilde{\sigma}^n D_{mn})\varphi, \\
&[I \otimes \tilde{\sigma}^m (w_m + A_m) + \tau^1 \otimes \tilde{\sigma}^m (p_m + B_m)]\varphi \\
&= (m_+ + \tau^1 \otimes I m_- - \mathrm{i} I \otimes \tilde{\sigma}^m \sigma^n C_{mn} - \mathrm{i}\tau^1 \otimes \tilde{\sigma}^m \sigma^n D_{mn})\bar{\chi}.
\end{aligned} \tag{1}$$

Here four-momenta w_m, p_m and mass parameters m_+, m_- are related to the quantities for the particles of the system by

$$w_m = \frac{1}{2}(p_{1m} + p_{2m}), \qquad p_m = \frac{1}{2}(p_{1m} - p_{2m}), \tag{2}$$

$$m_+ = \frac{1}{2}(m_1 + m_2), \qquad m_- = \frac{1}{2}(m_1 - m_2), \tag{3}$$

and the Lorentz-vector potentials A_m, B_m as well as the Lorentz-tensor ones C_{mn}, D_{mn} in general case are the functions of w_m, p_m and the relative coordinate $x_m = x_{1m} - x_{2m}$. The direct matrix products contain σ^m and $\tilde{\sigma}^m$ that are the 2×2 matrices from the one-particle Dirac equation in the Weyl notation, for which we use the conventional representation: $\sigma^m = (I, \boldsymbol{\tau})$, $\tilde{\sigma}^m = (I, -\boldsymbol{\tau})$ where I is the unit 2×2 matrix and $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$ are the Pauli matrices. Besides, the Minkowski metrics $h^{mn} = \mathrm{diag}(1, -1, -1, -1)$ is accepted.

Note that the first multipliers in the direct products originate due to the group extension and are essential for obtaining the two-particle interpretation of the problem. The above equation (1) should be supplemented with the subsidiary condition

$$(w^m p_m - m_+ m_-) \left(\begin{array}{c} \varphi \\ \bar{\chi} \end{array} \right) \equiv \frac{1}{4}(p_1^2 - p_2^2 - m_1^2 + m_2^2) \left(\begin{array}{c} \varphi \\ \bar{\chi} \end{array} \right) = 0, \tag{4}$$

which must guarantee that in the case when the interaction is absent the particles are on the mass shell and Eqs. (1) and (4) are reduced to the free Dirac and Klein-Gordon equations. These free equations are derived with decomposing the spinor wave functions into the projections

$$\varphi_\pm = \frac{1}{2}(1 \pm \tau^1 \otimes I)\varphi, \qquad \bar{\chi}_\pm = \frac{1}{2}(1 \pm \tau^1 \otimes I)\bar{\chi} \tag{5}$$

that are two-component $Sp(2,C)$ Weyl spinors. The projections labeled by plus (minus) sign describe the system in which the Dirac particle has mass m_1 (m_2) and the Klein-Gordon particle has mass m_2 (m_1).

Thus, the wave equation (1) supplemented with the subsidiary condition (4) describes two systems composed of the spin-1/2 and spin-0 particles, which differ from each other only in permutation of masses of the particles.

It is to be pointed that in the presence of the interaction the wave equation (1) and the subsidiary condition (4) must be compatible. The demand of the compatibility imposes the following restrictions on the shape of the potentials

$$\begin{aligned}
&\omega^m \pi_m + \pi_m \omega^m = 2 w_m p^m, \qquad \omega^m D_{mn} - D_{mn}\omega^m + C_{mn}\pi^m - \pi^m C_{mn} = 0, \\
&C_{mk} D^{mn} + D_{mk} C^{mn} = 0
\end{aligned} \tag{6}$$

where $\omega_m = w_m + A_m$, $\pi_m = p_m + B_m$. Moreover, the potentials must depend on the relative coordinate only through its transverse part

$$x_\perp^m = (h^{mn} - w^m w^n / w^2) x_n \tag{7}$$

with respect to the total four-momentum w_m, which is conserved and so can be treated as the eigenvalue rather than the operator.

It was shown [22] that in the presence of the only Lorentz-vector potentials the discussed approach reproduces the known equation by Królikowski [9], which is also derived by reducing the Bethe-Salpeter equation [11], and the equations of the relativistic quantum mechanics with constraints [15, 16]. In addition, being important for the study of mesonic atoms [24, 25], the problem of the description of the electromagnetic interaction in the system of a fermion with an anomalous magnetic moment and a boson was considered with the quasipotential [26] and Breit-type [24] equations. In our approach, the corresponding potentials for this problem are

$$\begin{aligned}
A_m &= \left(\left(1-\frac{2\mathcal{A}}{E}\right)^{1/2}-1\right)w_m, \\
B_m &= \left(\left(1-\frac{2\mathcal{A}}{E}\right)^{-1/2}-1\right)p_m+\frac{\mathrm{i}}{2E}\left(1-\frac{2\mathcal{A}}{E}\right)^{-3/2}\frac{\partial\mathcal{A}}{\partial x_\perp^m}, \\
C_{mn} &= 0, \qquad D_{mn}=\frac{k_1}{4m_1}\left(\frac{\partial A_n}{\partial x_\perp^m}-\frac{\partial A_m}{\partial x_\perp^n}+\frac{\partial B_n}{\partial x_\perp^m}-\frac{\partial B_m}{\partial x_\perp^n}\right) \qquad (8)
\end{aligned}$$

where $E = 2\sqrt{w^2}$ is the total energy, $\mathcal{A} = -\alpha/\sqrt{-x_\perp^2}$ is the Coulomb potential, k_1 and m_1 denote the anomalous magnetic moment and the mass of the fermion. With these potentials inserted, the wave equation (1) in the semirelativistic approximation coincides with that reported in Ref. [26].

Thus, the equation (1), derived with the extension of the $SL(2, C)$ group to the $Sp(4, C)$ one, together with the subsidiary condition (4) restores the results obtained in other approaches. However, including the Lorentz-tensor potentials in our equation, in addition to the Lorentz-vector ones, permits us to treat more wide range of problems concerning fermion-boson interactions and to construct the new oscillator model involved potentials of both these types of the Lorentz structure.

3 Exact oscillator-like solution

Now we will construct an oscillator model for the relativistic two-body system composed of the spin-1/2 and spin-0 particles, the word "oscillator" being taken to include all equations that can be transformed into the second-order equation containing quadratic in relative coordinate potentials. Such an oscillator-like second-order equation may arise from the first-order wave equation (1) if both the Lorentz-vector and -tensor potentials in Eq. (1) are linear in the relative coordinate.

However, in the case of the Lorentz-vector potentials, the purely linear in $x_\perp^m$ term can be removed by including this term into the phase factor of the wave function. Therefore, we suppose that the Lorentz-vector potentials involve not only the relative coordinate but also the relative four-momentum. A combination of these two quantities, having the structure of the multiplication of the orbital momentum with $x_\perp^m$, will give rise to the desired oscillator interaction. In order to avoid the Klein paradox, this Lorentz-vector oscillator potential must have the space-like rather than the time-like structure.

For constructing a Lorentz-tensor oscillator potential, we combine the linear in the relative coordinate term with the total four-momentum, which is the constant of motion, and set the potentials in Eq. (1) as follows

$$\begin{aligned}
&A_m = 0, \qquad && B_m = \lambda(x_\perp^2 p_{\perp m} - x_{\perp m}x_\perp^n p_{\perp n} - \mathrm{i}x_{\perp m}), \\
&C_{mn} = 0, \qquad && D_{mn} = \nu(w_m x_{\perp n} - w_n x_{\perp m})/2\sqrt{w^2}
\end{aligned} \qquad (9)$$

where λ and ν are constants with the dimensionality of the string tension, and the term with imaginary unit is introduced for the Hermiticity of the potential. Note that the oscillator potentials A_m and C_{mn} must be zero because, from Eqs. (1)–(3), they are transformed under the permutation of the particles like w_m ($w_m \to w_m$, $A_m \to A_m$, $C_{mn} \to C_{mn}$), but being linear in $x_\perp^m$, as the oscillator potentials, they must be transformed with the opposite sign.

The wave equation (1) with such an interaction proved to be exactly solvable. Since this equation describes two fermion-boson systems which differ from each other only in permutation of masses of

the particles, we accept that the system consists of a Dirac fermion with the mass m_1 and a Klein-Gordon boson with the mass m_2. To find the solution, we pass to the spinor projections φ_+ and $\bar{\chi}_+$, for which the wave equation (1) and the subsidiary condition (4), being written in the center-of-mass frame ($\mathbf{w}=0$), are reduced to the Dirac-like equation

$$\left(\frac{E}{2}+\frac{m_1^2-m_2^2}{2E}-m_1\right)\phi=\boldsymbol{\tau}\cdot(\boldsymbol{\pi}+\mathrm{i}\nu\mathbf{x})\psi,$$
$$\left(\frac{E}{2}+\frac{m_1^2-m_2^2}{2E}+m_1\right)\psi=\boldsymbol{\tau}\cdot(\boldsymbol{\pi}-\mathrm{i}\nu\mathbf{x})\phi \tag{10}$$

where $\phi=\bar{\chi}_++\varphi_+$, $\psi=\bar{\chi}_+-\varphi_+$, $\boldsymbol{\pi}=\mathbf{p}+\lambda(\mathbf{x}\times\mathbf{l}-\mathrm{i}\mathbf{x})$ and $\mathbf{l}=\mathbf{x}\times\mathbf{p}$ is the orbital momentum of the relative motion.

In the last equation, the total angular momentum of the relative motion, $\mathbf{j}=\mathbf{x}\times\mathbf{p}+\boldsymbol{\tau}/2$, is conserved. Then the spatial variables are separated and we can write

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\frac{G(r)}{r}\mathcal{Y}_{\kappa m}(\Omega)\\ \frac{\mathrm{i}F(r)}{r}\mathcal{Y}_{-\kappa m}(\Omega)\end{pmatrix} \tag{11}$$

where $\mathcal{Y}_{\kappa m}(\Omega)$ are the so-called spinor spherical harmonics [27] and the normalization of Dirac spinors imply that for the upper and lower radial functions we must have

$$\int_0^\infty (G^2(r)+F^2(r))dr=1, \tag{12}$$

so that $G(r)$ and $F(r)$ should be square-integrable functions.

The quantum number κ is related to the orbital quantum number l by the expression

$$\kappa=\begin{cases} l & =+(j+1/2)\,, \quad l=j+1/2 \quad (\kappa>0)\\ -(l+1) & =-(j+1/2)\,, \quad l=j-1/2 \quad (\kappa<0)\end{cases} \tag{13}$$

that completely determines the orbital and total angular momenta

$$l=|\kappa|+\frac{1}{2}\left(\frac{\kappa}{|\kappa|}-1\right), \qquad j=|\kappa|-\frac{1}{2} \tag{14}$$

and, hence, the parity $(-1)^l$.

Inserting (11) into (10), we obtain the set of radial equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}+(\nu-\lambda\kappa)r\right)G(r)-\left(\frac{E}{2}+\frac{m_1^2-m_2^2}{2E}+m_1\right)F(r)=0, \tag{15}$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}-(\nu-\lambda\kappa)r\right)F(r)+\left(\frac{E}{2}+\frac{m_1^2-m_2^2}{2E}-m_1\right)G(r)=0$$

which, with eliminating $F(r)$, can be converted into the oscillator-type equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}-(\nu-\lambda\kappa)^2r^2-(\nu-\lambda\kappa)(2\kappa-1)+\mathcal{E}\right)G(r)=0 \tag{16}$$

where

$$\mathcal{E}=\frac{E^2}{4}-\frac{m_1^2+m_2^2}{2}+\frac{(m_1^2-m_2^2)^2}{4E^2}. \tag{17}$$

This equation possesses the exact solution that reads as

$$G(r)=A\exp(-\frac{1}{2}a^2r^2)(a^2r^2)^{(l+1)/2}L_n^{l+1/2}(a^2r^2) \tag{18}$$

where $a=\sqrt{|\nu-\lambda\kappa|}$, $n=0,1,...$ denotes the radial quantum number and A is a normalization constant.

The lower radial function is obtained from Eqs. (15) with using the recursion relations for the generalized Laguerre polynomials [28].

For $\kappa > 0$ we have

$$\begin{aligned} F(r) = \quad & \frac{aEA}{(E+m_1)^2 - m_2^2} \exp(-\frac{1}{2}a^2r^2)(a^2r^2)^{l/2} \times \\ & \Big[(1+s)(n+l+1/2)L_n^{l-1/2}(a^2r^2) + (1-s)(n+1)L_{n+1}^{l-1/2}(a^2r^2)\Big] \end{aligned} \tag{19}$$

and, for $\kappa < 0$,

$$\begin{aligned} F(r) = \quad & \frac{aEA}{(E+m_1)^2 - m_2^2} \exp(-\frac{1}{2}a^2r^2)(a^2r^2)^{(l+2)/2} \times \\ & \Big[(1+s)L_n^{l+1/2}(a^2r^2) - 2L_n^{l+3/2}(a^2r^2)\Big] \end{aligned} \tag{20}$$

where $s = \mathrm{sgn}(\nu - \lambda\kappa)$.

Then the energy eigenvalues of Eq. (16) are defined by

$$\mathcal{E} = |\nu - \lambda\kappa|[4n + 2l + 3 + \mathrm{sgn}(\nu - \lambda\kappa)(2\kappa - 1)]. \tag{21}$$

Hence

$$E^2 = m_1^2 + m_2^2 + 2\mathcal{E} + 2\sqrt{(m_1^2 + \mathcal{E})(m_2^2 + \mathcal{E})} \tag{22}$$

describes the energy levels with quantum numbers (n, κ) or, using the standard spectroscopic notation, the energies of the states $n\,l_j$, because κ determines uniquely l and j.

It should be pointed that the appearance of the signum function in Eq. (21) imposes a restriction on the permitted values of the coupling constants λ and ν. For example, if $\nu > \lambda > 0$, there always exists a value of l such that for $\kappa = l$ we have $\mathrm{sgn}(\nu - \lambda\kappa) = +1$ whereas for $\kappa' = l' = l + 1$ we get $\mathrm{sgn}(\nu - \lambda\kappa') = -1$. Then from Eq. (21) it follows that for $l \le \nu/\lambda - 1/2$ the level $n\,(l+1)_{j+1}$ will be lower than the level $n\,l_j$. For avoiding this unphysical order of levels, it is sufficient to demand that either $|\lambda| \ge |\nu|$ or $\lambda = 0$. In the following, only the values of λ and ν which obey these conditions are considered.

4 Discussion

The derived energy spectrum of the fermion-boson system with the discussed oscillator interaction has proved to be essentially distinctive from that of the non-relativistic harmonic oscillator. The reason is that both the Lorentz-vector and -tensor potentials contribute to the strong spin-orbit coupling. In the expression (21) for the eigenenergies, the spin-orbit coupling is described by the term containing $(2\kappa - 1)$ [note that $\boldsymbol{\tau} \cdot \mathbf{l} = -(\kappa + 1)$]. This term breaks the $(2n + l)$-degeneracy, which is inherent in the spectrum of non-relativistic harmonic oscillator. Moreover, in Eq. (21) a common factor $|\nu - \lambda\kappa|$, playing the role of an effective oscillator frequency, is κ-dependent, too. This implies that the spectrum is not equidistant except for the case when the Lorentz-vector interaction is absent ($\lambda = 0$).

If $\lambda \ne 0$, the spacing of the energy levels depends on whether λ is positive or negative. For the case $\lambda > 0$, a typical spacing of the first energy levels is shown in Fig. 1 where the spectrum was evaluated with parameters $\lambda = 1$, $\nu = 0.1$, $m_1 = 1$ and $m_2 = 2$ in natural units. From Fig. 1 it can be seen that the ground-state levels with $n = 0$ and $l = j - 1/2 = 0, 1, ..$ are degenerate and correspond to the lowest energy eigenvalue $E = m_1 + m_2$. In contrast to this, the degeneracy does not occur in the case $\lambda < 0$, as seen from Fig. 2, in which the energy levels are evaluated using $\lambda = -1$, $\nu = 0.1$, $m_1 = 1$ and $m_2 = 2$.

To proceed, let us consider a particular case when $\lambda \ne 0$ and $\nu = 0$, i.e. when the only Lorentz-vector interaction is present. It should be noticed that in this case our model reduces to that offered within the framework of the relativistic quantum mechanics with constraints [15]. In Ref. [15], the subcase $\lambda > 0$ was considered and the corresponding energy spectrum was proved to display the parity-doubling phenomenon. However, we have found that in the other subcase ($\lambda < 0$) not only the exact oscillator-like solution exists, but also the parity doubling has a different meaning.

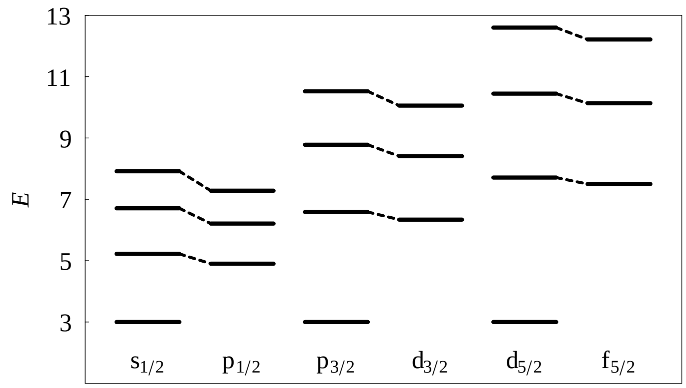

Figure 1: Energy levels for $\lambda = 1$, $\nu = 0.1$, $m_1 = 1$ and $m_2 = 2$ in natural units. The levels with the same j which become degenerate when $\nu = 0$ are connected by dashed lines.

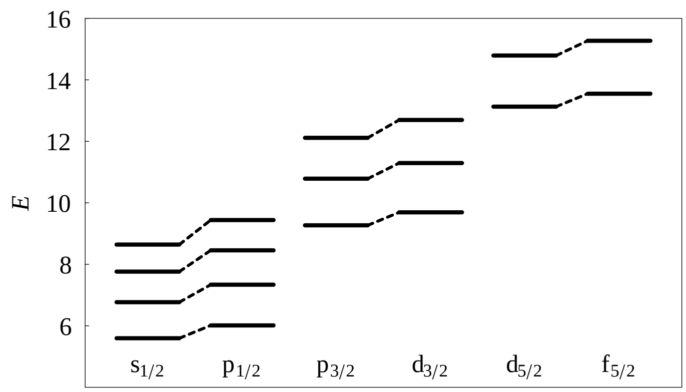

Figure 2: Energy levels for $\lambda = -1$, $\nu = 0.1$, $m_1 = 1$ and $m_2 = 2$ in natural units. The levels with the same j which become degenerate when $\nu = 0$ are connected by dashed lines.

For $\lambda > 0$ the parity doubling means that the state with quantum numbers $(n, j, l = j + 1/2)$ is degenerate with the state with the quantum numbers $(n + 1, j, l = j - 1/2)$, as can be checked by putting $\nu = 0$ in Eq. (21) [see also Fig. 1]. On the contrary, for $\lambda < 0$ the parity doubling reveals itself as the degeneracy of the states with $(n, j, l = j + 1/2)$ and $(n, j, l = j - 1/2)$ which correspond to the same value of n [see Fig. 2].

A few remarks concerning the parity doubling are in order. There exists reasonably strong experimental evidence that orbitally excited baryons occur in pairs of nearly degenerate states of opposite parity (for a review see [29]). In our oscillator model, the parity doubling of this kind takes place in the subcase $\lambda < 0$, whereas for $\lambda > 0$ the spectrum includes the degenerate ground states with $n = 0$ and $l = j - 1/2$ which are absent in the baryon spectra. In addition, the appearance of the parity doubling is closely related to the Lorentz-structure of the involved potentials. So, in the generalized Nambu-Jona-Lasinio model of hadrons the parity doubling occurs when the space-like Lorentz-vector potential dominates over the Lorentz-scalar one [30,31]. Analogously, in our approach the parity doubling appears when $\nu = 0$ and the potential becomes a pure spatial Lorentz vector.

Now let us turn to the second particular case, in which $\lambda = 0$, i.e., the pure Lorentz-tensor interaction is retained. In this case our oscillator-like solution can be treated as the generalization of the one-body Dirac oscillator model [4] to the fermion-boson case. Recall that the Dirac oscillator is obtained by replacing the momentum $\mathbf{p}$ in the Dirac equation by

$$\mathbf{p} \to \mathbf{p} - \mathrm{i} m\omega\beta\mathbf{x} \tag{23}$$

where β is the usual Dirac matrix and ω is the oscillator frequency. It is obvious that Eq. (10) for

the fermion-boson system transforms into the equation for the Dirac oscillator after inserting $\lambda = 0$, $\nu = m_1\omega$ and passing to the one-particle limit, in which the boson has a mass much larger than the fermion. Note that the passage to this one-particle limit does not affect the interaction terms in Eq. (10). Therefore, in the case $\lambda = 0$ the spectrum of the fermion-boson system will have the same order of the energy levels as the spectrum of the Dirac oscillator.

In conclusion, we have constructed the relativistic oscillator model for the two-body system consisted of the spin-1/2 fermion and the spin-0 boson interacting by virtue of the Lorentz-vector and Lorentz-tensor potentials. For this model, the analytical expressions for the wave functions and the energy eigenvalues have been obtained. The exact solubility of the oscillator model suggests it may be used as the first approximation in the study of more complicated systems in hadron and nuclear physics.

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References

[1] G. B. Smith, L. J. Tassie, *Ann. Phys. (N. Y.)* **65**, 352 (1971).

[2] R. Tegen, *Ann. Phys. (N. Y.)* **197**, 439 (1990).

[3] J. N. Ginocchio, *Phys. Rev. Lett.* **95**, 252501 (2005).

[4] M. Moshinsky, A. Szczepaniak, *J. Phys. A* **22**, L817 (1989).

[5] V. I. Kukulin, G. Loyola, M. Moshinsky, *Phys. Lett. A* **158**, 19 (1991).

[6] T. S. Chen, H. F. Lü, J. Meng, S. Q. Zhang, S. G. Zhou, *Chin. Phys. Lett.* **20**, 358 (2003).

[7] R. Lisboa, M. Malheiro, A. S. de Castro, P. Alberto, M. Fiolhais, *Phys. Rev. C* **69**, 024319 (2004).

[8] G. Breit, *Phys. Rev.* **34**, 553 (1929).

[9] W. Królikowski, *Acta Phys. Pol. B* **10**, 739 (1979).

[10] J. Bijtebier, J. Broekaert, *Nuovo Cim.* **105A**, 625 (1992).

[11] T. Tanaka, A. Suzuki, M. Kimura, *Z. Phys. A* **353**, 79 (1995).

[12] A. O. Barut, S. Komy, *Fortschr. Phys.* **33**, 309 (1985).

[13] W. Garczyński, M. Klimek, *Acta Phys. Pol. B* **20**, 755 (1989).

[14] M. J. King, F. Rohrlich, *Ann. Phys. (N. Y.)* **130**, 350 (1980).

[15] H. Sazdjian, *Phys. Rev. D* **33**, 3401, 3435 (1986).

[16] H. W. Crater, P. Van Alstine, *Phys. Rev. D* **36**, 3007 (1987).

[17] H. W. Crater, P. Van Alstine, *Phys. Rev. D* **70**, 034026 (2004).

[18] Ph. Droz-Vincent, *Annales Poincaré Phys. Theor.* **27**, 407 (1977).

[19] A. del Sol Mesa, M. Moshinsky, *J. Phys. A* **27**, 4685 (1994).

[20] M. Moshinsky, A. G. Nikitin, *Rev. Mex. Fis.* **50**, 66 (2005), arXiv: hep-ph/0502028.

[21] D. A. Kulikov, R. S. Tutik, A. P. Yaroshenko, *Phys. Lett. B* **644**, 311 (2007).

[22] D. A. Kulikov, R. S. Tutik, *Mod. Phys. Lett. A* **23**, 1829 (2008).

[23] R. Penrose, W. Rindler, *Spinors and Space-Time*, Cambridge University Press, 1984.

[24] N. G. Kelkar, M. Nowakowski, *Phys. Lett. B* **651**, 363 (2007).

[25] J. Gasser, V. E. Lyubovitskij, A. Rusetsky, *Phys. Rept.* **456**, 167 (2008).

[26] G. J. M. Austen, J. J. de Swart, *Phys. Rev. Lett.* **50**, 2039 (1983).

[27] W. Greiner, *Relativistic Quantum Mechanics*, third ed., Springer, Berlin, 2000.

[28] *Handbook of Mathematical Functions*, ed. by M. Abramowitz, I. Stegun, Dover Publications, New York, 1972.

[29] R. L. Jaffe, D. Pirjol, A. Scardicchio, *Phys. Rept.* **435**, 157 (2006).

[30] Yu. S. Kalashnikova, A. V. Nefediev, J. E. F. T. Ribeiro, *Phys. Rev. D* **72**, 034020 (2005).

[31] L. Ya. Glozman, A. V. Nefediev, J. E. F. T. Ribeiro, *Phys. Rev. D* **72**, 094002 (2005).